Indices

Fact (Laws of indices) —

$$a^m \times a^n = a^{m+n} \qquad a^m \div a^n = a^{m-n} \qquad (a^m)^n = a^{mn}$$

$$a^0 = 1 \qquad a^{-n} = \frac{1}{a^n} \qquad a^{\frac{1}{n}} = \sqrt[n]{a} \qquad a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$$

Example

Evaluate without a calculator:

$$81^{\frac{3}{4}} \qquad 32^{-\frac{2}{5}} \qquad \left(\frac{27}{8}\right)^{-\frac{2}{3}}$$

$81^{3/4} = 3^3 = 27$ $\quad 32^{-2/5} = 2^{-2} = \frac{1}{4}$ $\quad \left(\frac{8}{27}\right)^{2/3} = \frac{4}{9}$

Example

Simplify fully:

1. $\left(32x^5\right)^{-\frac{2}{5}}$
2. $x^2\left(4x^{-\frac{1}{2}}\right)^3$
3. $\dfrac{x-4x^3}{2x^2}$, written as the sum of powers of x

1. $32^{-2/5}x^{-2} = \dfrac{1}{4x^2}$
2. $x^2 \cdot 64x^{-3/2} = 64x^{1/2}$
3. $\dfrac{x}{2x^2} - \dfrac{4x^3}{2x^2} = \frac{1}{2}x^{-1} - 2x$

Index Equations

Example

Solve:

1. $4^x = 8^{x-1}$
2. Express 9^{3x+1} in the form 3^y, giving y in terms of x.
3. $\dfrac{x}{\sqrt[3]{x}} = 128$

1. $2^{2x} = 2^{3(x-1)} \implies 2x = 3x - 3 \implies x = 3$
2. $\left(3^2\right)^{3x+1} = 3^{6x+2}$: $y = 6x + 2$
3. $x^{2/3} = 128 \implies x = 128^{3/2} = \left(2^7\right)^{3/2} = 2^{21/2} = 2^{10}\sqrt{2} = 1024\sqrt{2}$

Textbook Exercises: SPS Course 1.8, Exercises 3B, 4 and 5

Surds

Fact —

$$\sqrt{ab} = \sqrt{a}\sqrt{b} \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \qquad \text{but } \sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$$

To **rationalise** a denominator, multiply by 1 in a useful form:

$$\frac{1}{\sqrt{a}} \times \frac{\sqrt{a}}{\sqrt{a}} \qquad \frac{1}{a+\sqrt{b}} \times \frac{a-\sqrt{b}}{a-\sqrt{b}}$$

Example

Simplify:

1. $\sqrt{48} - \dfrac{6}{\sqrt{3}}$
2. $(5 - \sqrt{8})(1 + \sqrt{2})$

1. $4\sqrt{3} - 2\sqrt{3} = 2\sqrt{3}$
2. $5 + 5\sqrt{2} - 2\sqrt{2} - 2 \cdot 2 = 1 + 3\sqrt{2}$ *(using $\sqrt{8} = 2\sqrt{2}$)*

Example

Simplify $\dfrac{7+\sqrt{5}}{\sqrt{5}-1}$, giving your answer in the form $a + b\sqrt{5}$.

$$\frac{(7+\sqrt{5})(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)} = \frac{7\sqrt{5}+7+5+\sqrt{5}}{4} = \frac{12+8\sqrt{5}}{4} = 3 + 2\sqrt{5}$$

Example (Edexcel C1)

Solve the equation

$$10 + x\sqrt{8} = \frac{6x}{\sqrt{2}}$$

giving your answer in the form $a\sqrt{b}$ where a and b are integers.

$\sqrt{8} = 2\sqrt{2}$ *and* $\dfrac{6x}{\sqrt{2}} = 3\sqrt{2}x$*:*

$$10 + 2\sqrt{2}x = 3\sqrt{2}x \implies 10 = \sqrt{2}x \implies x = \frac{10}{\sqrt{2}} = 5\sqrt{2}$$

Example

A rectangle has area $20\,\text{cm}^2$ and one side of length $(3 + \sqrt{5})$ cm. Find the length of the other side in the form $a + b\sqrt{5}$.

$$\frac{20}{3+\sqrt{5}} = \frac{20(3-\sqrt{5})}{9-5} = 5(3-\sqrt{5}) = 15 - 5\sqrt{5}$$

Textbook Exercises: SPS Course 1.7, Exercise 2 and Exam Questions 1.7.5